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Difference equations with special polynomials as solutions

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Abstract. In this paper, we introduce the difference equations of Bernoulli polynomials constructed using trigonometric functions and quantum numbers. Several types of difference equations have Bernoulli polynomials (QSB and QCB) as solutions and contain various properties.

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1. Introduction

This section briefly outlines the essential definitions and theorems required for understanding this study. For $q \in \mathbb{R} - \{1\}$, the q-number is defined as:

$$[n]_q = \frac{1-q^n}{1-q}.$$

In the definition of the q-number, it noted that $\lim_{q\to 1}[n]_q = n$, see [2], [3, [8]. Moreover, for $k \in \mathbb{Z}$, $[k]_q$ is referred to as a q-integer. The q-numbers introduced by Jackson ([3]) have led to expanded theories that intersect with established fields, see, [1], [2], [7], [8].

The q-Gaussian binomial coefficients ([**?**]) are defined as

$$\begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{[m]_q!}{[m-r]_q![r]_q!}.$$

Here, m and r denote non-negative integers.

Note that $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$ and $[0]_q! = 1$.

Definition 1.1. Let x be any complex numbers with $|x| < 1$. Then, two forms of q-exponential functions ([1], [2]) can be expressed as

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!},$$
$$E_q(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_q!}.$$

It is noted that $\lim_{q\to 1} e_q(x) = e^x$ and $e_q(x)E_q(-x) = 1$.

Definition 1.2. The q-derivative of a function f with respect to x is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \qquad \text{for} \quad x \neq 0,$$

and $D_q f(0) = f'(0)$, see, [6], [8].

We use the derivative with respect to x, y, and t, which are expressed as $D_{q,x}$, $D_{q,y}$, and $D_{q,t}$, respectively.

Definition 1.3. The classical function for the q-Bernoulli numbers and polynomials ([5], [7]) are

$$\sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{[n]_q!} = \frac{t}{e_q(t) - 1},$$
$$\sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{[n]_q!} = \frac{t}{e_q(t) - 1} e_q(tx), \quad \text{respectively.}$$

For $q \to 1$ in Definition 1.3., we can find the Bernoulli numbers B_n and polynomials $B_n(x)$.

In [5], the authors introduced new Bernoulli polynomials (sine Bernoulli polynomials and cosine Bernoulli polynomials) by replacing x with complex numbers and studied several properties thereof.

Definition 1.4. The generating function for the q-SINE Bernoulli (QSB) and q-COSINE Bernoulli (QCB) polynomials are

$$\sum_{n=0}^{\infty} B_{n,q}(x,y)\frac{t^n}{[n]_q!} = \frac{t}{e_q(t)-1}e_q(tx)\mathrm{SIN}_q(ty),$$
$$\sum_{n=0}^{\infty} B_{n,q}(x,y)\frac{t^n}{[n]_q!} = \frac{t}{e_q(t)-1}e_q(tx)\mathrm{COS}_q(ty),$$

respectively, see [5].

Theorem 1.5 [6]. *Let k be a non-negative integer. Then, the following relations can be formulated:*

$$\text{(i)} \quad S_{n-k,q}(x,y) = \frac{[n-k]_q!}{[n]_q!}D_{q,x}^{(k)}S_{n,q}(x,y).$$
$$\text{(ii)} \quad C_{n-k,q}(x,y) = \frac{[n-k]_q!}{[n]_q!}D_{q,x}^{(k)}C_{n,q}(x,y).$$

Theorem 1.6 [6]. *Let k be a non-negative integer. Then, the following is valid:*

$$\text{(i)} \quad D_{q,y}^{(k)}S_{n,q}(x,y) = \begin{cases} (-1)^{\frac{k}{2}}\dfrac{[n]_q!}{[n-k]_q!}S_{n-k,q}(x,q^k y), & \text{if } k \text{ is even,} \\ (-1)^{\frac{k-1}{2}}\dfrac{[n]_q!}{[n-k]_q!}C_{n-k,q}(x,q^k y), & \text{if } k \text{ is odd.} \end{cases}$$

$$\text{(ii)} \quad D_{q,y}^{(k)}C_{n,q}(x,y) = \begin{cases} (-1)^{\frac{k}{2}}\dfrac{[n]_q!}{[n-k]_q!}C_{n-k,q}(x,q^k y), & \text{if } k \text{ is even,} \\ (-1)^{\frac{k+1}{2}}\dfrac{[n]_q!}{[n-k]_q!}S_{n-k,q}(x,q^k y), & \text{if } k \text{ is odd.} \end{cases}$$

2. Several types of difference equations with QCB and QSB polynomials

In this Section, we use the Theorems 2.1. and 2.2. to verify the q-difference equations associated with QSB and QCB polynomials. The

q-difference equations that vary based on the variables are shown to have QSB and QCB polynomials as solutions.

Theorem 2.1. *For $k \in$ non-negative integer, we have the following relations with ${}_CB_{n,q}(x,y)$ and ${}_SB_{n,q}(x,y)$:*

$$\text{(i)} \quad D_{q,x}^{(k)}{}_CB_{n,q}(x,y) = \frac{[n]_q!}{[n-k]_q!}{}_CB_{n-k,q}(x,y),$$
$$\text{(ii)} \quad D_{q,x}^{(k)}{}_SB_{n,q}(x,y) = \frac{[n]_q!}{[n-k]_q!}{}_SB_{n-k,q}(x,y).$$

Proof. (i) Using the q-derivative in ${}_CB_{n,q}(x,y)$ about x, we get:

$$\begin{aligned} D_{q,x}^{(1)}\sum_{n=0}^{\infty}{}_CB_{n,q}(x,y)\frac{t^n}{[n]_q!} &= t\sum_{n=0}^{\infty}{}_CB_{n,q}(x,y)\frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty}[n]_q{}_CB_{n-1,q}(x,y)\frac{t^n}{[n]_q!}. \end{aligned} \tag{1}$$

After comparing the coefficients of t^n in Equation (1), we can formulate:

$$\begin{aligned} D_{q,x}^{(1)}{}_CB_{n,q}(x,y) &= [n]_q{}_CB_{n-1,q}(x,y) \\ &= \frac{[n]_q!}{[n-1]_q!}{}_CB_{n-1,q}(x,y). \end{aligned}$$

Via induction, we obtain Theorem 2.1 (i).

(ii) If we apply the proof of (i) of the Theorem 2.1 similarly to ${}_SB_{n,q}(x,y)$, we can derive (ii) of the theorem; hence, the proof process is omitted. □

Theorem 2.2. *Let k be a non-negative integer. Then, the following hold:*

$$\text{(ii)} \quad D_{q,y}^{(k)}{}_CB_{n,q}(x,y) = \begin{cases} (-1)^{\frac{k}{2}}\dfrac{[n]_q!}{[n-k]_q!}{}_CB_{n-k,q}(x,q^ky), & \text{if } k \text{ is even,} \\ \\ (-1)^{\frac{k+1}{2}}\dfrac{[n]_q!}{[n-k]_q!}{}_SB_{n-k,q}(x,q^ky), & \text{if } k \text{ is odd.} \end{cases}$$

$$\text{(i)} \quad D_{q,y}^{(k)}{}_SB_{n,q}(x,y) = \begin{cases} (-1)^{\frac{k}{2}} \dfrac{[n]_q!}{[n-k]_q!}{}_SB_{n-k,q}(x,q^ky), & \text{if } k \text{ is even}, \\ (-1)^{\frac{k-1}{2}} \dfrac{[n]_q!}{[n-k]_q!}{}_CB_{n-k,q}(x,q^ky), & \text{if } k \text{ is odd}. \end{cases}$$

Proof. (i) Applying the q-derivative in ${}_CB_{n,q}(x,y)$ with respect to y, we obtain

$$D_{q,y}^{(1)} \sum_{n=0}^{\infty} {}_CB_{n,q}(x,y)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} {}_SB_{n,q}(x,qy)\frac{t^{n+1}}{[n]_q!}$$
$$= \sum_{n=0}^{\infty} [n]_q {}_SB_{n,q}(x,qy)\frac{t^n}{[n]_q!}. \tag{2}$$

Using the coefficient comparison method and induction in (2), we can write:

$$D_{q,x}^{(1)}{}_CB_{n,q}(x,y) = [n]_q {}_SB_{n-1,q}(x,qy) = \frac{[n]_q!}{[n-1]_q!}{}_SB_{n-1,q}(x,qy),$$
$$D_{q,x}^{(2)}{}_CB_{n,q}(x,y) = -[n]_q[n-1]_q {}_CB_{n-2,q}(x,q^2y) = -\frac{[n]_q!}{[n-2]_q!}{}_CB_{n-2,q}(x,q^2y),$$
$$\vdots$$

to derive the desired result.

(ii) If we apply the proof process of (i) of Theorem 2.2 similarly to ${}_SB_{n,q}(x,y)$, we can derive (ii) of the theorem; hence, the proof process is omitted. □

Theorem 2.3. *(i) The q-difference equation of the form*

$$\frac{B_{n,q}}{[n]_q!}D_{q,x}^{(n)}S_{n,q}(x,y)$$
$$+ \frac{B_{n-1,q}}{[n-1]_q!}D_{q,x}^{(n-1)}S_{n,q}(x,y)$$
$$+ \frac{B_{n-2,q}}{[n-2]_q!}D_{q,x}^{(n-2)}S_{n,q}(x,y) + \cdots + \frac{B_{2,q}}{[2]_q!}D_{q,x}^{(2)}S_{n,q}(x,y)$$
$$+ B_{1,q}D_{q,x}^{(1)}S_{n,q}(x,y)$$
$$+ B_{0,q}S_{n,q}(x,y) - {}_SB_{n,q}(x,y) = 0$$

has ${}_{S}B_{n,q}(x,y)$ *as a solution.*

(ii) The polynomial ${}_{C}B_{n,q}(x,y)$ *is a solution of*

$$\begin{aligned}&\frac{B_{n,q}}{[n]_q!}D_{q,x}^{(n)}C_{n,q}(x,y)\\&+\frac{B_{n-1,q}}{[n-1]_q!}D_{q,x}^{(n-1)}C_{n,q}(x,y)\\&+\frac{B_{n-2,q}}{[n-2]_q!}D_{q,x}^{(n-2)}C_{n,q}(x,y)+\cdots+\frac{B_{2,q}}{[2]_q!}D_{q,x}^{(2)}C_{n,q}(x,y)\\&+B_{1,q}D_{q,x}^{(1)}C_{n,q}(x,y)\\&+B_{0,q}C_{n,q}(x,y)-{}_{C}B_{n,q}(x,y)=0.\end{aligned}$$

Proof. (i) Using the generating function of QSB polynomials, we find a relation for ${}_{S}B_{n,q}(x,y)$, $B_{n,q}$, and $S_{n,q}(x,y)$ as

$$\sum_{n=0}^{\infty}{}_{S}B_{n,q}(x,y)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q B_{k,q}S_{n-k,q}(x,y)\right)\frac{t^n}{[n]_q!}. \tag{3}$$

Comparing both sides of Equation (3) for t^n yields,

$${}_{S}B_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q B_{k,q}S_{n-k,q}(x,y). \tag{4}$$

If we replace Equation (4) with Theorem 1.5.(i), we can write

$${}_{S}B_{n,q}(x,y)=\sum_{k=0}^{n}\frac{B_{k,q}}{[k]_q!}D_{q,x}^{(k)}S_{n,q}(x,y). \tag{5}$$

We obtain the desired result by expanding the series in Equation (5).

(ii) Using a procedure similar to Equation (3) for the QCB polynomial, we can write:

$${}_{C}B_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q B_{k,q}C_{n-k,q}(x,y). \tag{6}$$

Using Theorem 1.5.(ii), Equation (6) becomes Equation (7):

$${}_{C}B_{n,q}(x,y)=\sum_{k=0}^{n}\frac{B_{k,q}}{[k]_q!}D_{q,x}^{(k)}C_{n,q}(x,y). \tag{7}$$

From Equation (7), we can derive Theorem 2.3. □

Corollary 2.4. *For $q \to 1$ in Theorem 2.3, the following holds:*

$$\text{(i)}\frac{B_n}{n!}\frac{d^n}{dx^n}S_n(x,y)+\frac{B_{n-1}}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}S_n(x,y)+\cdots+\frac{B_2}{2!}\frac{d^2}{dx^2}S_n(x,y)$$
$$+B_1\frac{d}{dx}S_n(x,y)+B_0S_n(x,y)-{}_SB_n(x,y)=0,$$
$$\text{(ii)}\frac{B_n}{n!}\frac{d^n}{dx^n}C_n(x,y)+\frac{B_{n-1}}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}C_n(x,y)+\cdots+\frac{B_2}{2!}\frac{d^2}{dx^2}C_n(x,y)$$
$$+B_1\frac{d}{dx}C_n(x,y)+B_0C_n(x,y)-{}_CB_n(x,y)=0.$$

Theorem 2.5. *Let n be a non-negative integer. Then, the q-difference equation below, for variable y, has ${}_SB_{n,q}(x,y)$ as the solution.*

(i) If n is a even number, then

$$\frac{(-1)^{\frac{n}{2}}B_{n,q}}{[n]_q!}D_{q,y}^{(n)}S_{n,q}(x,q^{-n}y)+\frac{(-1)^{\frac{n}{2}}B_{n-1,q}}{[n-1]_q!}D_{q,y}^{(n-1)}C_{n,q}(x,q^{1-n}y)$$
$$+\frac{(-1)^{\frac{n-2}{2}}B_{n-2,q}}{[n-2]_q!}D_{q,y}^{(n-2)}S_{n,q}(x,q^{2-n}y)+\cdots-\frac{B_{2,q}}{[2]_q!}D_{q,y}^{(2)}S_{n,q}(x,q^{-2}y)$$
$$-B_{1,q}D_{q,y}^{(1)}C_{n,q}(x,q^{-1}y)+B_{0,q}S_{n,q}(x,y)-{}_SB_{n,q}(x,y)=0.$$

(ii) If n is a odd number, then

$$\frac{(-1)^{\frac{n+1}{2}}B_{n,q}}{[n]_q!}D_{q,y}^{(n)}C_{n,q}(x,q^{-n}y)+\frac{(-1)^{\frac{n-1}{2}}B_{n-1,q}}{[n-1]_q!}D_{q,y}^{(n-1)}S_{n,q}(x,q^{1-n}y)$$
$$+\frac{(-1)^{\frac{n-1}{2}}B_{n-2,q}}{[n-2]_q!}D_{q,y}^{(n-2)}C_{n,q}(x,q^{2-n}y)+\cdots-\frac{B_{2,q}}{[2]_q!}D_{q,y}^{(2)}S_{n,q}(x,q^{-2}y)$$
$$-B_{1,q}D_{q,y}^{(1)}C_{n,q}(x,q^{-1}y)+B_{0,q}S_{n,q}(x,y)-{}_SB_{n,q}(x,y)=0.$$

Proof. In Theorem 1.6.(i), we can formulate

$$S_{n-k,q}(x,y)=\begin{cases}(-1)^{\frac{k}{2}}\dfrac{[n-k]_q!}{[n]_q!}D_{q,y}^{(k)}S_{n,q}(x,q^{-k}y), & \text{if } k \text{ is even},\\ (-1)^{\frac{k+1}{2}}\dfrac{[n-k]_q!}{[n]_q!}D_{q,y}^{(k)}C_{n,q}(x,q^{-k}y), & \text{if } k \text{ is odd}.\end{cases} \tag{8}$$

Applying Equation (8) in Equation (4), we can complete the proof of Theorem 2.5. □

Corollary 2.6. *Setting $q \to 1$ in Theorem 2.5, the following holds:*

(i) *If n is a even number, then*

$$\begin{aligned}&\frac{(-1)^{\frac{n}{2}}B_n}{n!}\frac{d^n}{dy^n}S_n(x,y)+\frac{(-1)^{\frac{n}{2}}B_{n-1}}{(n-1)!}\frac{d^{n-1}}{dy^{n-1}}C_n(x,y)\\&+\frac{(-1)^{\frac{n-2}{2}}B_{n-2}}{(n-2)!}\frac{d^{n-2}}{dy^{n-2}}S_n(x,y)+\cdots-\frac{B_2}{2!}\frac{d^2}{dy^2}S_n(x,y)\\&-B_1\frac{d}{dy}C_n(x,y)+B_0S_n(x,y)-{}_SB_n(x,y)=0.\end{aligned}$$

(ii) If n is a odd number, then

$$\begin{aligned}&\frac{(-1)^{\frac{n+1}{2}}B_n}{n!}\frac{d^n}{dy^n}C_n(x,y)+\frac{(-1)^{\frac{n-1}{2}}B_{n-1}}{(n-1)!}\frac{d^{n-1}}{dy^{n-1}}S_n(x,y)+\cdots\\&\cdots-\frac{B_2}{2!}\frac{d^2}{dy^2}S_n(x,y)-B_1\frac{d}{dy}C_n(x,y)+B_0S_n(x,y)-{}_SB_n(x,y)=0.\end{aligned}$$

Theorem 2.7. *For variable y, ${}_CB_{n,q}(x,y)$ is one of the following solutions of the q-difference equations: (i) If n is a even number, then*

$$\begin{aligned}&\frac{(-1)^{\frac{n}{2}}B_{n,q}}{[n]_q!}D_{q,y}^{(n)}C_{n,q}(x,q^{-n}y)+\frac{(-1)^{\frac{n-2}{2}}B_{n-1,q}}{[n-1]_q!}D_{q,y}^{(n-1)}S_{n,q}(x,q^{1-n}y)\\&+\frac{(-1)^{\frac{n-2}{2}}B_{n-2,q}}{[n-2]_q!}D_{q,y}^{(n-2)}C_{n,q}(x,q^{2-n}y)+\cdots-\frac{B_{2,q}}{[2]_q!}D_{q,y}^{(2)}C_{n,q}(x,q^{-2}y)\\&+B_{1,q}D_{q,y}^{(1)}S_{n,q}(x,q^{-1}y)+B_{0,q}C_{n,q}(x,y)-{}_CB_{n,q}(x,y)=0.\end{aligned}$$

(ii) If n is a odd number, then

$$\begin{aligned}&\frac{(-1)^{\frac{n-1}{2}}B_{n,q}}{[n]_q!}D_{q,y}^{(n)}S_{n,q}(x,q^{-n}y)+\frac{(-1)^{\frac{n}{2}}B_{n-1,q}}{[n-1]_q!}D_{q,y}^{(n-1)}C_{n,q}(x,q^{1-n}y)\\&+\frac{(-1)^{\frac{n-3}{2}}B_{n-2,q}}{[n-2]_q!}D_{q,y}^{(n-2)}S_{n,q}(x,q^{2-n}y)+\cdots-\frac{B_{2,q}}{[2]_q!}D_{q,y}^{(2)}C_{n,q}(x,q^{-2}y)\\&+B_{1,q}D_{q,y}^{(1)}S_{n,q}(x,q^{-1}y)+B_{0,q}C_{n,q}(x,y)-{}_CB_{n,q}(x,y)=0.\end{aligned}$$

Proof. In Theorem 1.6.(ii), it can be observed that

$$C_{n-k,q}(x,y)=\begin{cases}(-1)^{\frac{k}{2}}\dfrac{[n-k]_q!}{[n]_q!}D_{q,y}^{(k)}C_{n,q}(x,q^{-k}y), & \text{if } k \text{ is even},\\ \\ (-1)^{\frac{k-1}{2}}\dfrac{[n-k]_q!}{[n]_q!}D_{q,y}^{(k)}S_{n,q}(x,q^{-k}y), & \text{if } k \text{ is odd}.\end{cases} \tag{9}$$

Considering Equation (9) in Equation (6), we obtain the result of Theorem 2.7. □

Theorem 2.8. *For $e_q(t)\neq -1$, the QSB polynomial is one of the solutions of the following n-th order difference equation:*

$$\begin{aligned}&\frac{1}{[n]_q!}D_{q,x}^{(n)}{}_SB_{n,q}(x,y)+\frac{1}{[n-1]_q!}D_{q,x}^{(n-1)}{}_SB_{n,q}(x,y)\\&+\frac{1}{[n-2]_q!}D_{q,x}^{(n-2)}{}_SB_{n,q}(x,y)+\cdots+\frac{1}{[2]_q!}D_{q,x}^{(2)}{}_SB_{n,q}(x,y)\\&+D_{q,x}^{(1)}{}_SB_{n,q}(x,y)+2\left({}_SB_{n,q}(x,y)-S_{n,q}(x,y)\right)=0.\end{aligned}$$

Proof. If $e_q(t)\neq -1$ in the generating function of QSB polynomials, the following derivation is obtained:

$$\begin{aligned}&2\sum_{n=0}^{\infty}S_{n,q}(x,y)\frac{t^n}{[n]_q!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q{}_SB_{n-k,q}(x,y)+{}_SB_{n,q}(x,y)\right)\frac{t^n}{[n]_q!}.\end{aligned} \tag{10}$$

After comparing the series on both sides in Equation (10), we can write:

$$2S_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q{}_SB_{n-k,q}(x,y)+{}_SB_{n,q}(x,y). \tag{11}$$

If we substitute Theorem 2.1.(ii) into the right-hand side of Equation (10), we can formulate

$$\sum_{k=0}^{n}\frac{1}{[k]_q!}D_{q,x}^{(k)}{}_SB_{n,q}(x,y)+{}_SB_{n,q}(x,y)-2S_{n,q}(x,y)=0. \tag{12}$$

By expanding the finite series on the left-hand side of Equation (12), we obtain the desired result. □

Theorem 2.9. *The q-difference equation*

$$\frac{1}{[n]_q!}D_{q,x}^{(n)}{}_CB_{n,q}(x,y)+\frac{1}{[n-1]_q!}D_{q,x}^{(n-1)}{}_CB_{n,q}(x,y)+\cdots+D_{q,x}^{(1)}{}_CB_{n,q}(x,y)$$
$$+2\left({}_CB_{n,q}(x,y)-C_{n,q}(x,y)\right)=0$$

has ${}_CE_{n,q}(x,y)$ as the solution.

Proof. Similar to the procedure used for finding Equation (11) in Theorem 2.8, the relationship between ${}_CB_{n,q}(x,y)$ and $C_{n,q}(x,y)$ is:

$$2C_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q{}_CB_{n-k,q}(x,y)+{}_CB_{n,q}(x,y). \tag{13}$$

Substituting (ii) of Theorem 2.1. into the right-hand side of Equation (13), we obtain:

$$\sum_{k=0}^{n}\frac{1}{[k]_q!}D_{q,x}^{(k)}{}_CB_{n,q}(x,y)+{}_CB_{n,q}(x,y)-2C_{n,q}(x,y)=0. \tag{14}$$

Using Equation (14), we can finish the proof of Theorem 2.9. □

Corollary 2.10. *For $q\to 1$ in Theorems 2.8 and 2.9, the following holds:*

$$\text{(i)}\frac{1}{n!}\frac{d^n}{dx^n}{}_SB_n(x,y)+\frac{1}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}{}_SB_n(x,y)+\cdots$$
$$\cdots+\frac{1}{2!}\frac{d^2}{dx^2}{}_SB_n(x,y)+\frac{d}{dx}{}_SB_n(x,y)$$
$$+2\left({}_SB_n(x,y)-S_n(x,y)\right)=0.$$
$$\text{(ii)}\frac{1}{n!}\frac{d^n}{dx^n}{}_CB_n(x,y)+\frac{1}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}{}_CB_n(x,y)+\cdots$$
$$+\cdots\frac{1}{2!}\frac{d^2}{dx^2}{}_CB_n(x,y)+\frac{d}{dx}{}_CB_n(x,y)$$
$$+2\left({}_CB_n(x,y)-C_n(x,y)\right)=0.$$

3. Conclusion

We have identified several differential equations whose solutions are QCB polynomials or QSB polynomials. It was confirmed that differential equations appear in various ways depending on the variables, and in order to present mathematical modeling in the future, we need to further study various differential equations.

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